Double Diffusive Magneto-Convection Fluid Flow in a Strong Cross Magnetic Field With Uniform Surface Heat and Mass Flux

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In this study, magnetohydrodynamic natural convection boundary layer flow of an electrically conducting and viscous incompressible fluid along a heated vertical flat plate with uniform heat and mass flux in the presence of strong cross magnetic field has been investigated. Asymptotic solutions are obtained for small ($\ll 1$) and large ($\gg 1$) values of local Hartmann parameter, $\xi$, through regular perturbation method and matched asymptotic expansion technique, respectively. However, for all values of $\xi$ the boundary layer equations are transformed to a suitable form by using the free variable formulation (FVF) as well as the stream function formulation (SFF). The equations obtained through FVF are integrated via direct finite difference method together with Gaussian elimination technique while the others obtained through SFF are integrated numerically via Thomas algorithm. Discussion is carried out for fluids having small $Pr \ll 1$. The results obtained for small, large and all $\xi$ regimes are examined in terms of shear stress, $\tau_w$, rate of heat transfer, $q_a$, and rate of mass transfer, $m_a$, for important physical parameter. Attention has been given to the influence of Schmidt number, $Sc$, buoyancy ratio parameter, $N$ and local Hartmann parameter, $\xi$ on velocity, temperature and concentration distributions and noted that velocity and temperature of the fluid achieve their asymptotic profiles for $Sc \geq 10.0$.

[DOI: 10.1115/1.4007130]

Keywords: natural convection, cross field, magnetohydrodynamics, liquid metals, heat and mass flux

1 Introduction

The study of magnetohydrodynamic (MHD) viscous flows is important in industrial, technological, and geothermal applica-
tions, such as high-temperature plasmas, cooling of nuclear reactors, liquid metal fluids, MHD accelerators, and power generation systems. The term “crossed-fields” has been used to describe certain flows of electrically conduction fluids involving, in some undisturbed region, a uniform fluid stream and a uniform magnetic field making a nonzero angle with it. It is clear that there must be an electric field in such a region, directed perpendicular to both the stream velocity and magnetic-field vectors (see Sears [1]). Singh and Cowling [2] initially studied the boundary layer flow up a hot vertical plate in the presence of a uniform horizontal magnetic field normal to the plate. They considered the case when magnetic drag dominates viscous and inertial forces and concluded that regardless of the strength of magnetic field, there will always be a region near to the leading edge of the surface where electromagnetic force is negligible, but in the downstream regime this magnetic force is dominating. Subsequently, Riley [3] and Kuiken [4] reexamined the problem with a view to incorporate in the solution the inner viscous layer with in the downstream boundary layer which is appropriate if no slip boundary condition at the plate is considered. Their attempts to use the method of matched asymptotic expansion in terms of a nondimensional parameter in this region encountered difficulties associated with the asymptotic nature of solution in respect of an unknown location of the leading edge. First time, Wilks [5] reformulated the above problem in terms of coordinate expansion with respect to a nondimensional characteristic length that varies along the plate. Wilks [5] successfully obtained the complete numerical solutions which for the first time provides details of skin friction and heat transfer coefficients at all stations along the plate. Approximate solutions for upstream and downstream regions were obtained through series expansion method while Keller-box scheme was adopted to acquire full numerical solutions. Favorable comparison has been made between solutions obtained for the upstream region, downstream region and the intermediate region. Later, for the low Prandtl number fluids, appropriate for liquid metals, Wilks and Hunt [6] studied the problem in a strong cross magnetic field for free convection flow by examining the boundary constraint associated with uniform heat flux at the plate. This article also contains the solutions for small, large and all values of $\xi$ (where $\xi$ measures the local relative magnitudes of physical mechanisms at various stations). Near and far away from the leading edge of the plate solutions were expressed through series expansion method and Keller-box scheme was used in getting the full numerical solutions. It should be noted that none of the above studies discussed the mass transfer effects together with the strong cross magnetic field.

However, in nature and in many engineering applications there are many transport processes which are governed by the joint action of both thermal and mass diffusion. The engineering applications include chemical reactions in a reactor chamber, chemical vapor depositions of solid layers, combustion of atomized liquid fuels and dehydration operations in chemical and foundry plants (see Gebhart and Pera [7,8]). Further information on simultaneous heat and mass transfer in laminar free convection boundary layer flows over plates can be found in Refs. [9–12]. Recently, Palani and Srikanth [13] studied the conjugate effect of thermal and mass diffusion on MHD natural convection flow of an electrically conducting and viscous incompressible fluid past a semi-infinite vertical plate under the action of transversely applied uniform magnetic field. An unconditionally stable implicit finite difference scheme was hired to solve the nondimensional governing equations for the fluid having $Pr = 0.71$ (air) and $Pr = 7.0$ (water).

In processes such as drying, evaporation at the surface of a water body, energy transfer in a wet cooling tower and the flow in a desert cooler, heat and mass transfer occurs simultaneously. It is worth-
attention has been given to the study of double diffusive magnetoconvection flow of electrically conducting viscous incompressible fluid from a semi-infinite vertical plate with uniform surface heat and mass flux in the presence of strong cross magnetic field. The solutions are obtained for (i) small, (ii) large and (iii) all values of $\zeta$, a scaled distance from the leading edge. Specifically, for small $\zeta$ the unknown variables are expanded in terms of power series of $\zeta$ and then Runge–Kutta initial value solver together with Nachstein and Swigert [14] iterative technique is applied to obtain the solutions up to $O(\zeta^{4/5})$. Asymptotic solutions for large $\zeta$ are obtained analytically by adopting inverse coordinate expansion method. Finally, for entire values of $\zeta$ the governing equations are reduced to convenient form using the free variable formulation as well as the stream function formulation. The equations obtained by the former formulation have been simulated by the direct finite difference method together with Gaussian elimination technique and the latter one by Thomas algorithm. The results are obtained in terms of shear stress, $\tau_{xx}$, rate of heat transfer, $q_u$, and rate of mass transfer, $m_a$, and compared graphically with the results obtained for the entire $\zeta$ regime. Moreover, velocity, temperature, and mass concentration profiles are also shown for different values of Schmidt number, $Sc$, conjugate buoyancy parameter, $N$, small Prandtl number, $Pr$ and the local Hartmann parameter, $\xi$. For $N = 0$ (in the absence of species diffusion), some results are compared with that of Wilks and Hunt [6] which are found in good agreement.

2 Mathematical Formulation

Consider the magnetohydrodynamic natural convection boundary layer flow of an electrically conducting and viscous incompressible fluid along a heated vertical flat plate in the presence of strong cross magnetic field. In this problem, the ambient fluid temperature and species concentration are denoted by $T_\infty$ and $C_\infty$, respectively. It is further assumed that (i) the surface heat flux and mass flux are uniform, (ii) magnetic Reynolds number ($R_m = \mu_0 B_0 l_u L$) where $\mu_0$ is the magnetic permeability of the free space) is negligible, (iii) the cross-diffusion effects (i.e., Soret and Dufour effects) are also neglected compared with the direct effects, modeled by Fourier’s law and Fick’s law (see Gebhart and Pera [7,8]). In addition, all the thermophysical fluid properties are considered to be constant. The coordinate system and flow configuration of the problem are shown in Fig. 1.

Under the usual Boussinesq approximation the dimensionless boundary layer equations for conservation of mass, momentum, energy and species concentration can be written (following [3–8]) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\partial T}{\partial x}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\rho c_p} \frac{\partial q}{\partial x}$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{1}{\rho D} \frac{\partial q_c}{\partial x}$$

where $u$ and $v$ are the $\hat{x}$ and $\hat{y}$-components of the velocity field, respectively, in the momentum boundary layer; $T$ and $C$ are the temperature of the fluid in the thermal boundary layer and species concentration in the concentration boundary layer, respectively; $\nu$ is the kinematic viscosity of the fluid, $\sigma$ the electrical conductivity, $B_0$ the magnetic field normal to the plate, $\rho$ the density of the fluid, $g$ the acceleration due to gravity, $\kappa$ the thermal conductivity, $L$ the characteristic length of the plate, $q$ the uniform wall heat flux, $m$ the uniform wall mass flux, $\beta_T$ the coefficient of volume expansion, $\beta_c$ the coefficient of concentration expansion and $D$ the mass diffusion coefficient. The applied magnetic field or more correctly $\sigma B_0^2$ is supposed to be so large that the Lorentz and buoyancy forces are comparable and dominate the inertia force. Under these conditions the standard method of matched asymptotic expansions is applicable (see D’sa [15]).

In system of Eq. (1) the parameter $N$ measures the relative importance of chemical and thermal diffusion in causing the density difference which drives the flow. It should be noted that, for $N = 0$ the flow becomes unaffected by the species diffusion, whereas for extremely large value of $N$ the flow is not affected by thermal diffusion. Further, $N$ is positive for both effects combining to drive the flow and negative for the effects opposed (see Gebhart and Pera [7,8]). $Gr_L$ and $Gr_c$ are, respectively, the Grashof number due to the thermal and mass diffusivity and $Sc$ gives the ratio of momentum diffusivity to the mass diffusivity.

Now, we propose the methods of solution of the problem formulated in the set of Eq. (1–2).

3 Solution Methodologies

In this section, discussion has been carried out on various solution methodologies which are capitalized to solve the boundary

![Fig. 1 Coordinate system and physical model](image-url)
layer Eqs. (1)–(2) that govern the flow past a semi-infinite vertical flat plate with surface heat and mass flux in the strong cross magnetic field. Equations (1)–(2) are proposed to be obtained in (i) the upstream regime using regular perturbation method, (ii) the downstream regime using matched asymptotic solutions, and (iii) the entire upstream to downstream regime using the stream function formulation (SFF) and free variable formulation (FVF).

3.1 Solution Near the Leading Edge. It needs to be mentioned here that the boundary layer structure is expected to be similar near the leading edge of the plate. In this regime, the magnetic force is weaker in magnitude as compared to the buoyancy force. Therefore, in the vicinity of the leading edge of the plate the electrically conducting fluid feels the influence of buoyancy force. Hence, we introduce the following continuous transformations, which were first initiated by Wilks and Hunt [6] for the semi-infinite vertical flat plate. 

\[ \psi = x^{1/5}f(\eta, \xi), \quad \eta = x^{-1/5}y, \quad \xi = Mx, \quad T = x^{1/5}Q(\eta, \xi), \]

\[ C = x^{1/5} \phi(\eta, \xi), \]

where, \( \xi \) is termed as the local modified Hartmann parameter (or a characteristic coordinate, see Ref. [6]), which is small near the leading edge where the magnetic force is weaker in magnitude as compared to the buoyancy forces. Further, \( \psi \) is the stream function that satisfies the equation of continuity and is defined as

\[ u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \]

Thus, introducing, Eq. (4) into the set of Eqs. (1)–(2), we get the following nonsimilarity equations:

\[ f'''' + \frac{4}{5}f''' - \frac{3}{5}f'^2 - \xi^{2/5}f' - \frac{\theta + N\phi}{1 + N} = \xi \left( f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) \]

\[ \frac{1}{\Pr} f'''' + \frac{4}{5}f''' - f'' = \xi \left( f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) \]

\[ \frac{1}{Sc} f'''' + \frac{4}{5}f''' - f'' = \xi \left( f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) \]

The boundary conditions to be satisfied by the above equations are

\[ f(0, \xi) = f'(0, \xi) = 0, \quad f''(0, \xi) = \phi'(0, \xi) = 1 \]

\[ f''(\infty, \xi) \rightarrow 0, \quad 0'\left(\infty, \xi\right) \rightarrow 0, \quad \phi(\infty, \xi) \rightarrow 0 \]

Since \( \xi \) is small (\( \ll 1 \)), we now expand the functions \( f(\eta, \xi) \), \( 0(\eta, \xi) \) and \( \phi(\eta, \xi) \) in powers of \( \xi \) as given below

\[ f(\eta, \xi) = \sum_{i=0}^{\infty} \xi^{i/5}f_i(\eta), \quad 0(\eta, \xi) = \sum_{i=0}^{\infty} \xi^{i/5}0_i(\eta), \]

\[ \phi(\eta, \xi) = \sum_{i=0}^{\infty} \xi^{i/5} \phi_i(\eta) \]

Substituting the above series into Eqs. (6)–(7) and equating the coefficients upto \( O(\xi^{5/5}) \), we find three sets of coupled equations which are integrated sequentially with the help of Runge–Kutta initial value solver together with Nachtsheim and Swigert [14] iteration technique. Numerical results are thus obtained in terms of wall shear stress, \( \tau_w \), rate of heat transfer, \( q_w \), and rate of mass transfer, \( m_w \), from the following expressions:

\[ \tau_w = \xi^{2/5} \left( f''(0) + \xi^{2/5} f''(0) + \xi^{4/5} f''(0) + \cdots \right) \]

\[ q_w = -\left( \theta(0) + \xi^{2/5} \theta(0) + \xi^{4/5} \theta(0) + \cdots \right) \]

\[ m_w = -\xi^{2/5} \left( \phi(0) + \xi^{2/5} \phi(0) + \xi^{4/5} \phi(0) + \cdots \right) \]

Numerical values of \( \tau_w, q_w, \) and \( m_w \) obtained from the above expressions are shown graphically in Fig. 2.

Now we draw our attention on the possible asymptotic solutions for considerably large values of \( \xi \) by using matched asymptotic technique.

3.2 Asymptotic Solution for Large \( \xi \) (ASY). For the downstream region (large \( \xi \)), the boundary layer is developed primarily due to the magnetic forces parallel to the surface of the plate. It should be noted that far from the surface of the flat plate buoyancy force effects are less influential as compared to magnetic force. In this regard, Singh and Cowling [2], Riley [3], Kuiken [4] and Wilks [5] discussed the problem for the thermal diffusion only and elucidated the nature of solution for the region far away from the surface of the plate.

3.2.1 Outer Layer Region. Following Kuiken [4] the similarity equations for this region can be obtained by introduction the following scaled variables for \( \xi \gg 1 \), which refers to the effective balance between buoyancy and magnetic forces.

\[ f = \xi^{2/15} \hat{f}, \quad \hat{\eta} = \xi^{-2/15} \eta, \quad \hat{\theta} = \xi^{2/15} \hat{\theta}, \quad \hat{\phi} = \xi^{2/15} \hat{\phi} \]

Substituting Eq. (10) in to the system of Eqs. (6) to (7), we get the following:

\[ f'' + \frac{\hat{\theta}}{1 + N} = \frac{1}{\xi^{2/5}} \left( f'''' + \frac{2}{3}f''' - \frac{1}{3}f'' + \xi^{2/5} \frac{\partial f}{\partial \xi} \right) \]

\[ \frac{1}{\Pr} f'''' + \frac{2}{3}f''' - \frac{1}{3}f'' = \xi \left( f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) \]

\[ \frac{1}{Sc} f'''' + \frac{2}{3}f''' - \frac{1}{3}f'' = \xi \left( f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) \]

Fig. 2 (a) Variation of shear stress, (b) rate of heat transfer, and (c) rate of mass transfer with \( \xi \) for \( N = 0.0, 2.0, 5.0 \) while \( Pr = 0.054 \) and \( Sc = 10.0 \).
Corresponding boundary conditions are
\[ \begin{align*}
\hat f(0, \zeta) &= \hat f(0, \zeta) = 0, \quad \hat \theta(0, \zeta) = \hat \phi(0, \zeta) = 1 \\
\hat f(\infty, \zeta) &\to 0, \quad \hat \theta(\infty, \zeta) \to 0, \quad \hat \phi(\infty, \zeta) \to 0
\end{align*} \] (12)

The solution of the outer boundary layer region is established by using matched asymptotic expansion method. Thus following expansions are taken for the functions \( f, \theta \) and \( \phi \), respectively.
\[ \begin{align*}
f &= f_0 + \xi^{-1/3} \hat f_1 + \cdots, \quad \hat \theta = \hat \theta_0 + \xi^{-1/3} \hat \theta_1 + \cdots, \\
\hat \phi &= \hat \phi_0 + \xi^{-1/3} \hat \phi_1 + \cdots
\end{align*} \] (13)

Substituting Eq. (13) in the outer boundary layer, Eqs. (11)–(12), which leads to the set of ordinary differential equations for \( f_0, \theta_0 \) and \( \phi_0 \). Moreover, for considerably small \( \zeta \), the functions \( f_0(\zeta), \theta_0(\zeta) \) and \( \phi_0(\zeta) \) can be expressed in terms of power series as follows:
\[ \begin{align*}
f_0 &= \frac{z + N \delta}{1 + N} \eta - \frac{1}{2} \eta^2 + \frac{(z + N \delta)(Prz + NSc\delta)}{18(1 + N)^2} \eta^3 + \cdots \\
\theta_0 &= \frac{z + \eta - Pr(x + N \delta)}{6(1 + N)} \eta^2 - \frac{Pr(z + N \delta)}{18(1 + N)^2} \eta^3 + \cdots \\
\phi_0 &= \frac{\delta + \eta - Sc\delta(x + N \delta)}{6(1 + N)} \eta^2 - \frac{(x + N \delta) - (z + N \delta)}{18(1 + N)^2} \eta^3 + \cdots
\end{align*} \] (14)

where \( z \) and \( \delta \) are the missing conditions at \( \theta_0(0) \) and \( \phi_0(0) \), respectively, which are computed numerically for some fixed values of \( N, Pr \) and \( Sc \).

Expressions in Eq. (14) provides the ground over which various matching conditions should be evaluated for the solution of an inner boundary layer. In spite of the fact that above series solution attributed the dominant terms, i.e., the volume flow up the vertical plate and the heat and mass transfer across the plate, but it may be leaned on due to the violation of the no slip condition at the boundary. This is why inner boundary layer region is discovered so that this deficiency can be removed.

Further, functions \( f_1(\zeta), \theta_1(\zeta) \) and \( \phi_1(\zeta) \) can be obtained readily from the following linear coupled homogeneous system of equations.
\[ \begin{align*}
f_1' + \frac{\delta + \eta + N \phi}{1 + N} f_1 &= 0 \\
\frac{1}{Pr} \theta_1' + \frac{2}{3} \hat \theta_0 f_1 + \frac{1}{3} \hat \theta_0' f_1 - \frac{1}{3} f_1' \theta_0 &= 0 \\
\frac{1}{Sc} \phi_1' + \frac{2}{3} \hat \phi_0 f_1 + \frac{1}{3} \hat \phi_0' f_1 - \frac{1}{3} f_1' \phi_0 &= 0
\end{align*} \] (15)

which, to match with the inner solution should satisfy the following boundary conditions:
\[ \begin{align*}
f_1(0) &= \frac{z + N \delta}{1 + N}, \quad f_1(0) = -1, \quad \theta_1(0) = 1, \quad \phi_1(0) = 1
\end{align*} \] (16)

An appropriate solution for the functions \( f_1(\eta), \theta_1(\eta) \) and \( \phi_1(\eta) \) is found to be \( f_0(\eta), \theta_0(\eta) \) and \( \phi_0(\eta) \), respectively. Thus, from Eq. (13) solution for the outer layer expansion is found to be
\[ \begin{align*}
f &= f_0 + \xi^{-1/3} f_1 + \cdots, \quad \theta = \theta_0 + \xi^{-1/3} \theta_1 + \cdots, \\
\phi &= \phi_0 + \xi^{-1/3} \phi_1 + \cdots
\end{align*} \] (17)

where \( f_0, \theta_0, \phi_0 \) are given in Eq. (14).

Now we intend to obtain the governing equations for the inner boundary layer in the strong cross magnetic field.

3.2.2 Inner Layer Region. Riley [3] was the first who pointed out the inner boundary layer region near the surface of the vertical wall, so that no slip condition is well observed there. As suspected, viscous forces are present in the vicinity of the plate and favorably have the same order of magnitude as that of magnetic and buoyancy forces and in order to obtain the governing equations for this region, which are comparable with that of Eqs. (11)–(12), we take into account the following scaled variables:
\[ \begin{align*}
\hat f(\eta, \zeta) &= \xi^{-1/3} \hat f(\hat \eta, \xi), \quad \hat \eta = \xi^{-1/3} \eta, \quad \hat \theta(\eta, \zeta) = \hat \theta(\hat \eta, \xi), \\
\hat \phi(\eta, \zeta) &= \hat \phi(\hat \eta, \xi)
\end{align*} \] (18)

Introducing transformation (18) in Eqs. (11)–(12), we get the following governing equations for inner boundary layer region:
\[ \begin{align*}
f'' - \frac{\theta + N \phi}{1 + N} + \frac{1}{3 \xi^{2/3}} \left[ f'' - f'' \right] &= \xi^{\frac{1}{3}} \left( \frac{\phi''}{\xi} - \frac{\theta''}{\xi} \right) \\
\frac{1}{Pr} \theta'' + \frac{1}{3 \xi^{2/3}} \left[ f'' - f'' \right] &= \xi^{\frac{1}{3}} \left( \frac{\phi''}{\xi} - \frac{\theta''}{\xi} \right) \\
\frac{1}{Sc} \phi'' + \frac{1}{3 \xi^{2/3}} \left[ f'' - f'' \right] &= \xi^{\frac{1}{3}} \left( \frac{\phi''}{\xi} - \frac{\theta''}{\xi} \right)
\end{align*} \] (19)

Boundary conditions are
\[ \begin{align*}
\hat f(0, \zeta) &= \hat f(0, \zeta) = 0, \quad \hat \theta(0, \zeta) = \hat \phi(0, \zeta) = \xi^{-1/3}, \\
\hat f(\infty, \zeta) &\to 0, \quad \hat \theta(\infty, \zeta) \to 0, \quad \hat \phi(\infty, \zeta) \to 0
\end{align*} \] (20)

Here we take the following expansions for the functions \( f, \theta \) and \( \phi \).
\[ \begin{align*}
f &= f_0 + \xi^{-1/3} f_1 + \cdots, \quad \theta = \theta_0 + \xi^{-1/3} \theta_1 + \cdots, \\
\phi &= \phi_0 + \xi^{-1/3} \phi_1 + \cdots
\end{align*} \] (21)

Substitution of expansions given in Eq. (21) in the inner boundary layer, Eqs. (19)–(20) leads to the set of ordinary differential equations, which are solved up to \( O(\xi^{-1/3}) \) and the solution is obtained of the form:
\[ \begin{align*}
f &= \left( \frac{z + N \delta}{1 + N} \right) \left( 1 - \eta - \hat n - e^{-\hat n} \right) + \xi^{-1/3} \left( 1 - \eta - \frac{\hat n^2}{2} - e^{-\hat n} \right) + \cdots \\
\theta &= \eta + \xi^{-1/3} \left( 1 + \hat n \right) + \cdots \\
\phi &= \delta + \xi^{-1/3} \left( 1 + \hat n \right) + \cdots
\end{align*} \] (22)

Now we are in a position to present the composite form of the solution.

3.2.3 Composite Solution. Following Van Dyke [16], we utilize the outer and inner expansions, Eqs. (17) and (22), respectively, and taking the inner of outer solution, Eq. (17) to express the uniformly valid solution in the following composite form:
\[ \begin{align*}
f &\sim \left( \frac{z + N \delta}{1 + N} \right) \xi^{-1/3} \left( 1 - \xi^{1/3} \eta - e^{-\xi^{1/3} \eta} \right) + \xi^{-2/3} \left( 1 - \xi^{1/3} \eta - \frac{\xi^{2/3} \eta^2}{2} - e^{-\xi^{1/3} \eta} \right) \\
\theta &\sim \eta + \xi^{-1/3} \left( 1 + \eta \right) + \cdots \\
\phi &\sim \delta + \xi^{-1/3} \left( 1 + \eta \right) + \cdots
\end{align*} \] (23)

The asymptotic representation of shear stress, \( \tau_n \), heat transfer rate, \( q_n \), and mass transfer rate, \( m_n \), take the following form:
\[ \tau_w \approx -\frac{2^{1/3}}{3} \left[ \frac{x + N \delta}{1 + N} \right] + 2^{1/3}, \quad q_w \approx -\frac{\zeta^{-1/3}}{3} \left[ \frac{x + \zeta^{-1/3}}{1 + \zeta^{-1/3}} \right]. \]  
\[ m_w \approx -\frac{\zeta^{-1/3}}{\delta + \zeta^{-1/3}}. \]  

Some asymptotic values of \( \tau_w, q_w \), and \( m_w \) obtained from the above relations are compared graphically with that obtained from the other methods in Fig. 2.

Now, we give our attention to find the solution of problem posed above for entire \( \zeta \) regime, i.e., for \( 0 < \zeta < \infty \). In this regard we have adopted two formulations, namely, the stream function formulation and the free variable formulation.

### 3.3 Numerical Solutions for Entire \( \zeta \) Regime

In this section, we introduce the stream function formulation (SFF) and the free variable formulation (FVF), which reduce the boundary layer, Eqs. (1)–(2) in to more convenient form for integration in the entire range of \( \zeta \) (i.e., \( 0 < \zeta < \infty \)).

#### 3.3.1 Stream Function Formulation

Following Wilks and Hunt [6], we compare the transformations given in Eqs. (4) and (10) and find the transformations given below

\[ f = (1 + \zeta)^{-2/15} \varphi, \quad \theta = (1 + \zeta)^{2/15} \Phi, \quad \phi = (1 + \zeta)^{2/15} \Phi, \]

\[ \eta = (1 + \zeta)^{1/15} \eta. \]

One can see that these transformations conveniently switch from one system to the other and facilitate successful integration over all \( \zeta \). Thus, introducing Eq. (25) in Eqs. (6) to (7) we get

\[ \frac{\rho w + 2(6 + 5 \zeta) \varphi^{(0)}}{15(1 + \zeta)} f \frac{\partial f}{\partial \zeta} - \frac{9 + 5 \zeta}{15(1 + \zeta)} f^2 \varphi^{(2)} - \frac{1}{15} \left[ 1 + \frac{N \varphi}{1 + N} \right] \frac{\partial \Phi}{\partial \zeta} - \frac{1}{15} \left[ \frac{\partial \varphi}{\partial \zeta} - \frac{\partial \Phi}{\partial \zeta} \right] + \frac{f^2 \varphi^{(0)}}{15(1 + \zeta)} f \frac{\partial f}{\partial \zeta} - \frac{3 + 5 \zeta}{15(1 + \zeta)} f^2 \varphi^{(2)} - \frac{1}{15} \left[ 1 + \frac{N \varphi}{1 + N} \right] \frac{\partial \Phi}{\partial \zeta} - \frac{1}{15} \left[ \frac{\partial \varphi}{\partial \zeta} - \frac{\partial \Phi}{\partial \zeta} \right] \right] \]

\[ = \frac{1}{15} \left[ \frac{\partial \varphi}{\partial \zeta} - \frac{\partial \Phi}{\partial \zeta} \right]. \]

The boundary conditions to be satisfied are

\[ f(0, \zeta) = f(0, \zeta) = 0, \quad \theta(0, \zeta) = \Phi(0, \zeta) = 1 \]

\[ f(\infty, \zeta) \rightarrow 0, \quad \frac{\partial f}{\partial \zeta} \rightarrow 0, \quad \frac{\partial \Phi}{\partial \zeta} \rightarrow 0. \]

The above set of equations may be utilized for complete numerical integration in the boundary layer region and provides the switch between the upstream and the downstream regimes. It is worth mentioning that in the absence of conjugate buoyancy parameter (i.e., for \( N = 0 \)), Eqs. (26) and (27) reduces to that investigated by Wilks and Hunt [6].

To find the solution of the above set of nonlinear parabolic partial differential equations, we assume that \( \beta = \beta^* \); this reduces the above set of Eqs. (26) and (27) into a second-order system of partial differential equations. We discretize the reduced second order system of equations for numerical scheme by replacing the partial derivatives with difference formulas, which leads to a system of tridiagonal algebraic equations for the momentum, energy and concentration equations. Thomas algorithm (tridiagonal solver) is used to evaluate the coupled system of equations for the unknowns \( f, \theta \) and \( \phi \). The computation has been started at \( \zeta = 0.0 \), and then marched up to \( \zeta = 100.0 \) using the step size \( \Delta \zeta = 0.025 \). At every \( \zeta \) station, the computations are iterated until the difference of the results of two successive iterations become less or equal to \( 10^{-6} \). In order to get accurate results we have compared the results at different grid size in \( \eta \) direction and reached at the conclusion to chose \( \Delta \eta = 0.01 \). The maximum value of \( \eta \) is taken to be 60.0. It is noteworthy that very recently the method just discussed has been used successfully by Siddiqua and Hosssain [17].

The important physical quantities like wall shear stress, \( \tau_w \), heat transfer rate, \( q_w \), and mass transfer rate, \( m_w \) can be calculated as \( f, \theta, \phi \) and their derivatives are evaluated at each \( \zeta \) step. Below are the expressions for these quantities, respectively.

\[ \tau_w = \zeta^{1/5} (1 + \zeta)^{-2/15} f(0, \zeta), \quad q_w = -\frac{\zeta^{-1/5}}{\delta(0, \zeta)} f \phi(0, \zeta), \]

\[ m_w = -\frac{\zeta^{-1/5}}{\phi(0, \zeta)} f \phi(0, \zeta) \]

It can be seen from the relations given in Eq. (28) that the wall shear stress and the rate of heat and mass transfer vary with local Hartmann parameter, \( \zeta \). Some numerical values thus obtained for \( \tau_w, q_w, m_w \), are entered in Table 1.

#### 3.3.2 Free Variable Formulation (FVF)

Free variable formulation (FVF) is initiated in the Eqs. (1) to (2) before employing the numerical scheme for the solution of the problem. For this, consider the following continuous transformations:

\[ u = x^{1/5} (1 + \zeta)^{-2/15} U, \quad v = x^{-1/5} (1 + \zeta)^{-2/15} V, \]

\[ Y = x^{-1/5} (1 + \zeta)^{-2/15}, \quad T = x^{1/5} (1 + \zeta)^{2/15}. \]

### Table 1 Numerical values of shear stress, rate of heat transfer and rate of mass transfer with \( \text{Pr} = 0.054, \ N = 5.0 \) and \( \text{Sc} = 10.0 \) and 20.0

<table>
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<th>FVF</th>
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<th>SFF</th>
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<th>SFF</th>
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<th>SFF</th>
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<th>SFF</th>
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<tbody>
<tr>
<td>( \tau_w )</td>
<td>( q_w )</td>
<td>( m_w )</td>
<td>( \tau_w )</td>
<td>( q_w )</td>
<td>( m_w )</td>
<td>( \tau_w )</td>
<td>( q_w )</td>
<td>( m_w )</td>
<td>( \tau_w )</td>
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<tr>
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<td>0.2624</td>
<td>0.2624</td>
<td>2.5810</td>
<td>2.5810</td>
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<td>0.4050</td>
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<td>0.2617</td>
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</table>
Using the transformations given in Eq. (29) into the Eqs. (1) to (2), one obtains
\[
\frac{(9 + 5\xi)}{15(1 + \xi)} U - \frac{3 + 5\xi}{15(1 + \xi)} \frac{\partial U}{\partial \xi} + \xi \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \xi} = 0
\]
\[
\frac{(9 + 5\xi)}{15(1 + \xi)} U^2 + \left(3 + 5\xi\right) \frac{\partial U}{\partial \xi} + \xi U \frac{\partial U}{\partial \xi}\]
\[
= \frac{\partial^2 U}{\partial \xi^2} - \xi^2(1 + \xi)^{1/15} U + (1 + \xi)^{2/15} \Theta + NV\]
\[
\frac{(3 + 5\xi)}{15(1 + \xi)} U \Theta + \left(3 + 5\xi\right) \frac{\partial \Theta}{\partial \xi} + \xi U \frac{\partial \Theta}{\partial \xi} = 1 \frac{\partial^2 \Theta}{\partial \xi^2} - \Theta \frac{\partial \Theta}{\partial \xi}
\]
\[
\frac{(3 + 5\xi)}{15(1 + \xi)} U \Phi + \left(3 + 5\xi\right) \frac{\partial \Phi}{\partial \xi} + \xi U \frac{\partial \Phi}{\partial \xi} = 1 \frac{\partial^2 \Phi}{\partial \xi^2} - \Phi \frac{\partial \Phi}{\partial \xi}
\]
(30)

The boundary conditions to be satisfied are
\[
U(0, \xi) = V(0, \xi) = 0, \quad \left(\frac{\partial \Theta}{\partial \xi}\right)_{y=0} = \left(\frac{\partial \Phi}{\partial \xi}\right)_{y=0} = 1
\]
U(\infty, \xi) \rightarrow 0, \quad \Theta(\infty, \xi) \rightarrow 0, \quad \Phi(\infty, \xi) \rightarrow 0
\]
(31)

The parabolic partial differential equations obtained in Eqs. (30) to (31) have been integrated numerically with the aid of direct finite difference method along with the Gaussian elimination technique. The computation is started at \(\xi = 0.0\) and then marches downstream implicitly. Here, 0.025 and 0.01 are the choices for the \(\xi\) and \(Y\) grids, respectively. For a given value of \(\xi\), the iterative procedure is stopped when the difference in computing the velocity, temperature, and concentration with the next iteration is \(< 10^{-6}\). Very recently, this method has been used successfully by Siddiqua et al. [17–19] to integrate coupled system of equations.

Finally, the shearing stress, \(\tau_w\), rate of heat transfer, \(q_w\), and rate of mass transfer, \(m_w\), are obtained from the following relations, respectively.

\[\tau_w = \xi^{2/5}(1 + \xi)^{-6/15} \frac{\partial U}{\partial y} \bigg|_{y=0}, \quad q_w = -\frac{\xi^{-1/5}(1 + \xi)^{-2/15}}{\Theta(0, \xi)}
\]
\[m_w = -\frac{\xi^{-1/5}(1 + \xi)^{-2/15}}{\Phi(0, \xi)}
\]
(32)

A comparison between the FVF and SFF is carried out in Table 1, which shows excellent agreement between these two methods. This comparison clearly validates that the numerical scheme used here are quite accurate. Further, it should be noted that all the results are obtained for fluids with Prandtl number, \(Pr = 0.054\).

## 4 Results and Discussion

In this article, investigation is made on the magnetohydrodynamic free convection flow of electrically conducting and viscous incompressible fluid along a heated vertical plate in a strong cross magnetic field with uniform surface heat and mass flux. For the case of upstream region, where \(\xi\) is assumed to be small enough, solutions are obtained by adopting the regular perturbation method. However, for considerably large \(\xi\) matched asymptotic technique of Wilks and Hunt [6] has been used. Furthermore, for all values of \(\xi\) the boundary layer equations are transformed to a suitable form by using FVF and SFF. The equations obtained through FVF are integrated via direct finite difference method in connection with Gaussian elimination technique whereas the others obtained through SFF are integrated numerically via Thomas algorithm.

It is worth mentioning that Wilks and Hunt [6] studied the same problem in the absence of mass diffusion, i.e., when buoyancy ratio parameter has no effects on the flow (\(N = 0.0\)). The present numerical results are compared with them in tabular form through Tables 2–3 and found in excellent agreement. In their analysis, Wilks and Hunt [6] aimed to investigate the influence of uniform surface heat flux on the boundary layer flow. They handled the problem numerically by using Keller box method for entire region where as series solution method is used to handle the problem in the vicinity of the plate. They firmly emphasized on the fact that for sufficiently large streamwise coordinate, \(\xi\), inverse coordinate

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expansion technique is appropriate in order to achieve consistency. Numerical values of the coefficients of skin friction \( f''_w(0) \), \( f''_w(0) \) and \( h''_w(0) \) and heat transfer \( \theta'_w(0) \), \( \theta'_w(0) \) and \( \theta'_w(0) \) thus obtained for \( N = 0.0 \) are entered in Tables 2–3 and compared with that of Wilks and Hunt [6], for several values of Pr. Comparison admits excellent compatibility qualitatively between the present analysis and work done by Wilks and Hunt [6].

Likewise, we present the results in terms of skin-friction, \( \tau_w \), rate of heat transfer, \( q_w \), and rate of mass transfer, \( m_w \), for multiple buoyancy parameter, \( N \) against locally distributed Hartmann parameter \( \xi \). Velocity, temperature and concentration profiles are drawn as well for different values of the physical parameters such as \( \xi, N, \text{Sc and Pr} \).

4.1 Effect of Physical Parameter \( N \) on Shear Stress, Rate of Heat and Mass Transfer. In Fig. 2, the influence of buoyancy ratio parameter, \( N \), on shear stress, \( \tau_w \), heat transfer rate, \( q_w \), and rate of mass transfer, \( m_w \), for multiple buoyancy parameter, \( N \) against locally distributed Hartmann parameter \( \xi \). Velocity, temperature and concentration profiles are drawn as well for different values of the physical parameters such as \( \xi, N, \text{Sc and Pr} \).

The effect of streamwise coordinate, \( \xi \), is exhibited in Fig. 3 for \( \xi = 0.0, 1.0, 3.0, 6.0, 10.0 \), and \( 20.0 \) while other physical parameters are \( N = 5.0, \text{Sc} = 10.0 \) and \( \text{Pr} = 0.054 \). It is inferred from the figure that application of magnetic field results in the considerable reduction in the flow velocity while temperature and species concentration profile increases owing to the increase in the streamwise coordinate, \( \xi \). Here it is observed that in the vicinity of the plate velocity of the fluid shoots up sharply and after attaining its maxima it settles down to its asymptotic values. The maxima of each curve reduces because of the intense magnetic field effects, which establishes the fact that velocity decreases effectively. As \( \xi \) increases the maxima moved towards the origin sharply, indicating that intense magnetic field causes the fluid to accelerate rapidly. It is also inferred from the Figs. 3(a)–3(b) that momentum and species concentration boundary layer thicknesses are notably reduced whereas thermal boundary layer thickness enhances for the increasing values of streamwise coordinate, \( \xi \). One can observe that with in the boundary layer velocity profile becomes 29.63% weaker as \( \xi \) increases from 0 to 1, similarly relative reduction in the flow velocity when \( \xi \) increases from 0–3, 0–6, 0–10 and 0–20 is respectively found to be 39.99%, 46.78%, 51.65% and 57.82%. However, temperature distribution is enhanced by the factors 20.32%, 28.63%, 35.46%, 41.45% and 50.97% when streamwise coordinate, \( \xi \), moves from 0–1, 0–3, 0–6, 0–10 and 0–20, respectively. Similarly, we have found that level of concentration of foreign mass is increased as well and measurement shows that increment in the values of \( \xi \) from 0–1, 0–3, 0–6, 0–10 and 0–20, the percentage increase in concentration distribution is 8.92%, 12.36%, 14.94%, 17.04% and 20.23%, respectively.

Now, the influence of Schmidt number, Sc, on velocity, temperature and concentration distributions is shown graphically against \( y \) in Fig. 4 for Sc = 1.0, 5.0, 10.0 and \( \text{Pr} = 0.054 \), \( N = 5.0 \) and \( \xi = 1.0 \). Figures 4(a)–4(c) show that velocity and species concentration of the fluid decreases extensively for influential species diffusion parameter, (Schmidt number) \( \text{Sc} \) and we get asymptotic velocity profile for \( \text{Sc} \geq 20.0 \). However, with in the boundary layer temperature of the fluid increases and achieve its asymptotic profile for the Schmidt number \( \text{Sc} \geq 20.0 \). It is readily observed that thickness of momentum, thermal and species concentration boundary

\[ \begin{align*}
\text{Fig. 3} & \quad (a) \text{Velocity, (b) temperature, and (c) concentration profiles for } \xi = 0.0, 1.0, 3.0, 6.0, 10.0, 20.0 \text{ while } \text{Pr} = 0.054, \\
\text{Sc} = 10.0 \text{ and } N = 5.0
\end{align*} \]

\[ \begin{align*}
\text{Fig. 4} & \quad (a) \text{Velocity, (b) temperature, and (c) concentration profiles for } \text{Sc} = 1.0, 5.0, 20.0 \text{ while } \text{Pr} = 0.054, \\
\xi = 1.0 \text{ and } N = 5.0
\end{align*} \]
layers decreases effectively and settle down to their asymptotic values when Schmidt number Sc increases from 1.0 to 20.0.

The effects of buoyancy ratio parameter, \( N \), is exhibited in Fig. 5 for the wide range of \( N = 0.0, 1.0, 2.0, 5.0, 7.0, 10.0 \) while other physical parameters are \( Sc = 10.0 \), \( \zeta = 1.0 \) and \( Pr = 0.054 \). One can see that buoyancy ratio parameter, \( N \), which gives the relative effects of buoyancy force that arise from mass diffusion and the thermal diffusion, has significant effect on velocity, temperature and species concentration profiles. Here, velocity of the fluid decreases extensively whereas temperature and concentration distributions become more influential inside the boundary layer. The thickness of momentum, thermal and level of mass diffusion boundary layers increases for higher values of buoyancy ration parameter, which clearly indicates that parameter \( N \) plays a vital role in the development of the flow field. Such effects are significant, especially in chemical industries, where combine effect of thermal and species diffusion is required. One can observe that with in the boundary layer velocity profile becomes 28.19% weaker as \( \zeta \) increases from 0 to 1, similarly relative reduction in the flow velocity when \( \zeta \) increases from 0–2, 0–5, 0–7 and 0–10 is, respectively, found to be 41.22%, 58.76%, 64.44% and 69.94%. However, temperature distribution in the flow region gets stronger by the factors 26.57%, 44.87%, 81.83% and 121.02% when streamwise coordinate, \( \xi \), moves from 0–1, 0–2, 0–5, 0–7 and 0–10, respectively. Similarly, we have found that level of species diffusion is increased as well and measurement shows that increment in the values of \( \zeta \) from 0–1, 0–2, 0–5, 0–7 and 0–10, the percentage increase in concentration distribution is 19.87%, 32.64%, 55.85%, 65.70% and 76.47%, respectively.

Lastly, in Fig. 6 the variation due to several values of Prandtl number, \( Pr \) is noted on velocity, temperature and concentration distributions for \( N = 5.0, \zeta = 1.0 \) and \( Sc = 10.0 \). It can be viewed that velocity and temperature of the fluid decreases considerably while concentration profile increases as \( Pr \) enhances from 0.001 to 0.1. Increase in the values of \( Pr \) actually leads to decrease in thermal conductivity of the fluid, which reduces the frictional forces. Due to this, notable reduction is depicted in the velocity and temperature profiles. Further, it can be seen that momentum, thermal and concentration boundary layer thickness are much affected as Prandtl number becomes large. However in the present investigation, it is noted that for \( Pr \geq 0.054 \) momentum, thermal and concentration boundary layer thicknesses happen to overlap each other at \( y = 2.0 \) and no further variation is recorded.

5 Conclusions

In this paper, consideration has been given to the conjugate effect of heat and mass transfer on the two-dimensional free convection flow of an electrically conducting viscous incompressible fluid over a semi-infinite vertical flat plate with uniform heat and mass flux. The influence of magnetic field has also been considered. It is worthy to mention that present analysis has been done particularly for those fluids for which \( Pr \ll 1 \). For entire range of streamwise coordinate, \( \xi \), the governing equations are reduced to parabolic partial differential equations using SFF and FVF which are then integrated by employing the implicit finite difference method and the direct finite difference method. However, for slightly small values of streamwise coordinate, \( \xi \), problem is solved with regular perturbation method whereas asymptotic solutions are obtained for larger values of \( \xi \) by using matched asymptotic technique. Discussion has been carried out on the numerical results obtained in terms of shear stress, \( \tau \), heat transfer rate, \( q \), and mass transfer rate, \( mw \), for several physically important parameters, such as, buoyancy ratio parameter, \( N \), Schmidt number, \( Sc \), streamwise coordinate, \( \xi \), and Prandtl number, \( Pr \). Velocity, temperature and species concentration profiles are also drawn and analyzed critically in the presence of intense magnetic field. It is observed from the analysis that wall shear stress, heat transfer rate and mass transfer rate becomes lesser in magnitude owing to the increase in the buoyancy ratio parameter \( N \). Moreover, velocity, temperature and mass concentration profiles drawn for different values of Schmidt number, \( Sc \), shows that with in the boundary layer velocity and species concentration of the fluid achieve their asymptotic profiles for \( Sc \geq 10.0 \).

Nomenclature

- \( B_0 \) = magnetic field
- \( C \) = species concentration in the boundary layer

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\( C_w \) = species concentration at the surface  
\( C_\infty \) = species concentration of the ambient fluid  
\( C \) = dimensionless concentration  
\( D \) = diffusion coefficient  
\( g \) = acceleration due to gravity  
\( Gr_L \) = Grashof number for thermal diffusion  
\( Gr_M \) = Grashof number for mass diffusion  
\( M \) = magnetic parameter  
\( m \) = uniform wall mass flux  
\( N \) = buoyancy ratio parameter  
\( Pr \) = Prandtl number  
\( q_w \) = dimensionless heat transfer rate  
\( q \) = uniform wall heat flux  
\( Sc \) = Schmidt number  
\( T \) = temperature of the fluid in the boundary layer  
\( T_w \) = temperature at the surface  
\( T_\infty \) = temperature of the ambient fluid  
\( T^* \) = dimensionless temperature  
\( u, v \) = dimensionless fluid velocities in the \( x \)- and \( y \)-direction, respectively  
\( \hat{x}, \hat{y} \) = dimensional Cartesian coordinates  
\( x, y \) = dimensionless Cartesian coordinates  

Greek Symbols  
\( \zeta \) = streamwise coordinate  
\( \tau_w \) = dimensionless shear-stress  
\( \psi \) = stream function  
\( \kappa \) = thermal conductivity  
\( \beta_T \) = volumetric coefficient of thermal expansion  
\( \beta_\rho \) = volumetric coefficient of expansion with concentration  
\( \mu \) = dynamical viscosity  
\( \nu \) = kinematic viscosity  
\( \rho \) = density of the fluid  
\( \sigma \) = electrical conductivity

Subscripts  
\( w \) = wall condition  
\( \infty \) = ambient condition

References